Table 1 Gas phase decay lengths for both bulk and wall damping in a tube of 5-cm radius

	P = 1.1 atm		P = 30.7 atm	
f, cps	L_b, cm	L_w, cm	L_b, cm	L_w, cm
10^3	3.26×10^6	1.42×10^3	9.09×10^7	7.50×10^3
5×10^3	1.30×10^5	6.33×10^2	3.64×10^6	3.35×10^3
10^4	3.26×10^4	4.47×10^2	9.09×10^5	2.36×10^3
2×10^4	8.12×10^3	3.17×10^2	2.26×10^5	1.68×10^3

energy through viscous interaction with the particles is given by

$$dE/E = -3kVH''dx = -4kaNH''\pi a^2 dx \qquad (1)$$

$$H'' = \frac{\{12\}\{1 + \beta a\}\{[\beta a]^2 + [\delta/(1-\delta)][(\beta a)^2 + \frac{3}{2}\beta a]\}}{16[\beta a]^4 + 48[\delta/(1-\delta)][(\beta a)^4 + \frac{3}{2}(\beta a)^3] + 81[\delta/(1-\delta)]^2[\frac{4}{9}(\beta a)^4 + \frac{4}{3}(\beta a)^3 + 2(\beta a)^2 + 2(\beta a) + 1]} \qquad (2)$$

Equation (2), which is obtained from the results of Refs. 2–4, holds for $\beta a \ll 1$. When $\delta \ll \beta a$ and higher-order terms are dropped, Eq. (2) agrees with the result of Ref. 8. In the foregoing equations, $k = \omega/c$, $\beta = (\omega/2\nu)^{1/2}$, $\delta = \rho_0/\rho_1$, and V is the volume of solid particles per unit volume of total mixture, medium, and solid particles.

The decay length L_p over which the acoustic energy flux is reduced to $1/e$ of its original value is given by

$$L_p = 1/3kVH'' \qquad (3)$$

The viscous and thermal damping by the pure gas phase, without particles, consists of damping both in the bulk of the gas and at the walls of the tube. For the case of a plane wave traveling axially along a tube of radius R, the two damping constants are given by Refs. 5–7:

$$\sigma_b = \left[\frac{2\pi^2\mu}{\gamma c}\right]\left[\frac{4}{3} + \left(\frac{\gamma-1}{\gamma}\right)\left(\frac{\lambda}{\mu c_v}\right)\right]\left[\frac{f^2}{P_0}\right] \qquad (4)$$

$$\sigma_w = \left[\frac{(\mu\pi)^{1/2}}{R}\right]\left[\left(\frac{1}{\gamma}\right)^{1/2} + \left(\frac{\lambda}{\mu c_v}\right)^{1/2}\left(\frac{\gamma-1}{\gamma}\right)\right]\left[\frac{f}{P_0}\right]^{1/2} \qquad (5)$$

and the decay length is given by

$$L_b = 1/2\sigma_b \qquad L_w = 1/2\sigma_w \qquad (6)$$

In the case where all three types of damping are present and acting independently, the combined damping constant is given by

$$\frac{1}{L} = \frac{1}{L_p} + \frac{1}{L_b} + \frac{1}{L_w} \qquad (7)$$

Equations (2–5) were used to calculate the decay lengths L_p, L_b, L_w for a hypothetical propellant that contained 10% Al by weight. It was assumed that all of the aluminium was converted to liquid Al_2O_3.

The combustion gases were assumed to have the properties of CO at 3000°K. Thus, the physical constants had the following values: $c_v = 0.227$ cal/g-°K, $c = 1.12 \times 10^5$ cm/sec, $\lambda = 2.6 \times 10^{-4}$ cal/sec-cm-°K, $\gamma = 1.4$, $\mu = 7.75 \times 10^{-4}$ poise, $\rho_1 = 3.5$ g/cm³, $\rho_0 = 1.14 \times 10^{-4}$ g/cm³ per atmosphere pressure, $V = 0.233\ (\rho_0/\rho_1) = 0.0665\ \rho_0$.

The calculation was made for pressures of 1.1 and 30.7 atm. The results for L_p are shown in Fig. 1, whereas those for L_w and L_b (for R = 5 cm), are shown in Table 1.

It is noteworthy that the particle damping is relatively insensitive to chamber pressure, whereas the wall damping is inversely proportional to the square root of the chamber pressure. Thus, over the range of critical particle sizes where particle damping far exceeds wall damping, L will be insensitive to pressure level.

The maximum particle damping occurs in the size range 1 to 10 μ, which is a range that is available in commercial metal powders. Powders of smaller particle size are difficult to obtain, and most propellant systems probably operate with greater than the optimum particle size. Thus, the analysis indicates the desirability of adding the powders in the condition of the finest granulation that is obtainable. As would be expected, the damping increases as the quantity of additive increases [Eq. (3)].

The analysis takes no account of the chemical reactivity of the additives. This might profoundly alter their damping characteristics. The particle size of the products might, for example, be considerably different from that of the additive itself. Also, the temperature nonequilibrium between the particles and the gas and the jetting and spinning of the burning particles will probably change their damping properties.

Conclusions

The combustion products from metal additives in solid propellants can greatly increase the acoustic damping constant of the combustion gases. The particle sizes for which this effect is most pronounced are in the 1- to 10-μ range, which is about the minimum size range in which commercial metal powders are available. When the damping is predominately caused by particles in the combustion gases, it is insensitive to chamber pressure.

In Ref. 9, Horton and McGie have presented results of their calculations of the acoustic damping constant for propellent gases that contain 2% of Al_2O_3 particles. This analysis is part of a larger analysis of their experimental results, and their viewpoint is somewhat different than that taken in this note.

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Elliptic Elements in Terms of Small Increments of Position and Velocity Components

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Nomenclature

Elliptic elements

a = semimajor axis
L = mean longitude, measured from ξ axis

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p = η' component of a vector of length e, in the direction of the perifocus
q = ξ component of the same vector
W_ξ = $\sin\Omega \sin i$, the ξ component of the unit vector in the direction of the angular momentum
W_η = $-\cos\Omega \sin i$, the η' component of the same vector
Other symbols are defined in the text, as they are used.

Introduction

IT is well known that the relative motion of a point with respect to another point moving in a circular orbit in a spherical gravitational field can be expressed in a simple manner if the relative distance is small in relation to the orbital radius. The point of reference may be called the nominal point, moving in the nominal orbit.

In this note the problem of relative motion in close orbits is generalized for the case in which the nominal orbit is an ellipse. If nondimensional variables are introduced for the relative distances and the true anomaly is used as the independent variable (instead of the time), it is shown that the solution may be developed in powers of the nominal orbit's eccentricity. In essence this solution has the same simple form as that of the circular case.

In order to compare two perturbation solutions of a particular three-body problem, one in rectangular coordinates and one in elliptic elements, the author recently required expressions for the changes in the elliptic elements in terms of small changes in relative positions and velocities with respect to a nominal orbit. The main purpose of this note is to point out the interesting relationship between these changes in elements and the integration constants of the equations of relative motion.

Elliptic Elements in Terms of Relative Position and Velocity

Let units be chosen so that for the nominal orbit, the semimajor axis $a = 1$; also, let the gravitational constant $k^2 = 1$. The position of P is given by (ξ,η,ζ) in the relative coordinate system attached to the nominal point P_0 with the ξ axis in the direction of the nominal radius, the η axis perpendicular to ξ in the direction of orbital motion, and the ζ axis completing a right-handed coordinate system. The relative coordinate system moves with P_0 along the nominal orbit, at any instant rotating with angular velocity $\dot{v}$.

The elliptic elements of the orbit of P are to be expressed in terms of the components of relative position and velocity. The inertial position vector of P expressed in the rotating ξ,η',ζ coordinate system (where the η' axis is parallel to the η axis and goes through the center of attraction) is

$$\mathbf{R} = \begin{bmatrix} r_0 + \xi \\ \eta \\ \zeta \end{bmatrix} \tag{1}$$

Similarly the velocity vector of P is

$$\mathbf{V} = \begin{bmatrix} \dot{r}_0 + \dot{\xi} \\ \dot{\eta} \\ \dot{\zeta} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{v}_0 \end{bmatrix} \times \begin{bmatrix} r_0 + \xi \\ \eta \\ \zeta \end{bmatrix} = \begin{bmatrix} \dot{r}_0 + \dot{\xi} - \eta\dot{v}_0 \\ r_0\dot{v}_0 + \xi\dot{v}_0 + \dot{\eta} \\ \dot{\zeta} \end{bmatrix} \tag{2}$$

Here and in what follows, the subscript 0 refers to the nominal orbit.

If now nondimensional coordinates are introduced by

$$x = \xi/r_0 \qquad y = \eta/r_0 \qquad z = \zeta/r_0 \tag{3}$$

and the true anomaly of P_0 on the reference orbit is used as the independent variable instead of the time, one has

$$\mathbf{R} = r_0\begin{bmatrix} 1 + x \\ y \\ z \end{bmatrix} \qquad \mathbf{V} = r_0\dot{v}_0\begin{bmatrix} x' - y + (1 + x)\epsilon \\ 1 + x + y' + \epsilon y \\ z' + \epsilon z \end{bmatrix} \tag{4}$$

where

$$' \equiv d/dv$$

$$\epsilon = r_0'/r_0 = (e_0 r_0/h_0^2) \sin v_0$$

These expressions are now to be substituted in the relations between elements and coordinates, specialized for low-eccentricity orbits (see, for instance, Ref. 1). For small relative position and velocity vectors, the resulting formulas may be developed in powers of the components $(x,y,z,\dot{x},\dot{y},\dot{z})$. In the following this development has been carried out to include all the quadratic terms in $x, \ldots \dot{z}$. The elliptic elements are chosen as indicated in the Nomenclature.

This choice of elements is in part motivated by the desire to obtain expressions for orbits with small eccentricity and small inclination. More importantly, the motivation is found in the simple and elegant results that are obtained. These results can be connected immediately with the expressions for relative motion of two particles in close orbits. These expressions are well known for the case in which the reference point moves in a circular orbit and will be derived here for an elliptical reference orbit.

If X,Y,Z are the components of $\mathbf{R}$ and $\dot{X},\dot{Y},\dot{Z}$ are the components of $\mathbf{V}$, the components of the angular momentum vector $\mathbf{h}$ are found to be

$$h_x = Y\dot{Z} - Z\dot{Y} = h_0(-z - zx + yz' - zy')$$

$$h_y = Z\dot{X} - X\dot{Z} = h_0(-z' - yz + zx' - xz')$$

$$h_z = X\dot{Y} - Y\dot{X} = h_0(1 + 2x + y' + x^2 + y^2 + xy' - yx')$$

where $h_0 = r_0^2\dot{v}_0$, the angular momentum of the nominal orbit. The magnitude of the angular momentum is thus:

$$h = (h_x^2 + h_y^2 + h_z^2)^{1/2} = h_0(1 + 2x + y' + x^2 + y^2 + \tfrac{1}{2}z^2 + \tfrac{1}{2}\dot{z}^2 + xy' - yx')$$

The radial velocity $\dot{R}$ is

$$\dot{R} = \frac{\mathbf{R}\cdot\dot{\mathbf{R}}}{|\mathbf{R}|} = r_0\dot{v}_0(x' + \epsilon + \epsilon x + yy' + zz')$$

so that

$$e \sin v = \dot{R}h = e_0 \sin v_0 + (h_0^2/r_0)(x' + 2xx' + x'y' + yy' + zz' + 3\epsilon x + \epsilon y') \tag{5}$$

Also,

$$e \cos v = (h^2/r) - 1 = e_0 \cos v_0 + (h_0^2/r_0)(3x + 2y' + 3x^2 + \tfrac{3}{2}y^2 + \tfrac{1}{2}z^2 + 4xy' + y'^2 - 2yx' + z'^2) \tag{6}$$

The eccentricity follows from (5) and (6):

$$e^2 = e_0^2 + 2e_0x' \sin v_0 + 2e_0(3x + 2y') \cos v_0 + x'^2 + (3x + 2y')^2$$

and, with this and the angular momentum, the semimajor axis is

$$a = h^2/(1 - e^2) = 1 + 4x + 2y' + 15x^2 + 2y^2 + z^2 + 5y'^2 + z'^2 + 18xy' - 2yx' + x'^2 + 2e_0x' \sin v_0 + 2e_0(3x + 2y') \cos v_0 \tag{7}$$

In the development of this expression and in all the following, the nominal eccentricity e_0 has been treated as being of the same order as the components of relative position and velocity.

Now the vector $\mathbf{W}$ is defined as the unit vector in the direction of $\mathbf{h}$, the vector $\mathbf{U}$ as the unit vector in the direction of $\mathbf{R}$, and the vector $\mathbf{V}$ by $\mathbf{V} = \mathbf{W} \times \mathbf{U}$.

The components of the vector $\mathbf{a}$ of magnitude e and in the direction of the perifocus follow from the vectors $\mathbf{U}$ and $\mathbf{V}$ in combination with Eqs. (5) and (6). The expression h_0^2/r_0 has been replaced by $(1 + e_0 \cos v_0)$, and p_0 and q_0 are defined by $p_0 = -e_0 \sin v_0$ and $q_0 = e_0 \cos v_0$. The components of **a** are thus

$$\left.\begin{aligned} a_x &= U_x e \cos v - V_x e \sin v = q_0 + 3x + 2y' + 4xy' - x'y + 3x^2 + \tfrac{3}{2}y^2 + \tfrac{1}{2}z^2 + y'^2 + z'^2 + 3xq_0 + 2y'q_0 - yp_0 \\ a_y &= U_y e \cos v - V_y e \sin v = p_0 - x' + 3xy + yy' - 2xx' - x'y' - zz' - 3\epsilon x - \epsilon y' + yq_0 - x'q_0 \\ a_z &= U_z e \cos v - V_z e \sin v = q_0 z + 3xz + 2y'z + p_0 z' - x'z' \end{aligned}\right\} \quad (8)$$

The eccentric anomaly is best introduced in combination with the true anomaly:

$$(v - E) = \tan^{-1}\left(\frac{\sin(v-E)}{\cos(v-E)}\right) = \tan^{-1}(e \sin v) \times \left(\frac{1 + e\cos v + (1-e^2)^{1/2}}{1 + e\cos v + (1-e^2)^{1/2} + e\cos v(1-e^2)^{1/2} - (e^2 \sin^2 v)}\right)$$

Since only second-order terms are required, this is reduced quite easily to

$$(v - E) = e \sin v(1 - \tfrac{1}{2}e \cos v)$$

so that, with Eqs. (5) and (6) one has

$$(v - E) = (v_0 - E_0) + x' + \tfrac{1}{2}xx' + yy' + zz' + 3\epsilon x + \epsilon y' + \tfrac{1}{2}q_0 x' + \tfrac{3}{2}p_0 x + p_0 y' \quad (9)$$

The term $(e \sin E)$ in Kepler's equation is

$$e \sin E = \frac{e \sin v(1-e^2)^{1/2}}{1 + e \cos v} = e \sin v(1 - e \cos v)$$

since no terms of higher than second order are required. Thus, again with Eqs. (5) and (6),

$$e \sin E = e_0 \sin E_0 + x' - xx' - x'y' + yy' + zz' + 3\epsilon x + \epsilon y' + 3p_0 x + 2p_0 y' \quad (10)$$

Equations (9) and (10) now may be used in Kepler's equation to give the equation of the center:

$$(v - M) = (v - E) + e \sin E = (v_0 - M_0) + 2x' - \tfrac{1}{2}xx' - x'y' + 2yy' + 2zz' + 6\epsilon x + 2\epsilon y' + \tfrac{1}{2}q_0 x' + (9/2)p_0 x + 3p_0 y' \quad (11)$$

The true longitude l measured from the ξ axis is

$$l = \tan^{-1}\left(\frac{U_y - V_x}{U_x + V_y}\right) = \tan^{-1}\left(\frac{y(1-x) + \tfrac{1}{2}zz'}{1 - \tfrac{1}{2}y^2 - \tfrac{1}{2}z'^2}\right) = y - xy + \tfrac{1}{2}zz' \quad (12)$$

Finally, the mean longitude is found from Eqs. (11) and (12):

$$L = l - (v - M) = L_0 + y - 2x' - xy + \tfrac{1}{2}xx' + x'y' - 2yy' - \tfrac{3}{2}zz' - 6\epsilon x - 2\epsilon y' - \tfrac{1}{2}q_0 x' - \tfrac{9}{2}p_0 x - 3p_0 y' \quad (13)$$

Expressions for all the elements have now been obtained. Using only the first-order results, for the sake of simplicity, they are as follows:

$$\left.\begin{aligned} &\text{From Eq. (7)} \quad a = 1 + 2(2x + y') \\ &\text{From Eq. (13)} \quad L = L_0 + (y - 2x') \\ &\text{From Eq. (8)} \quad p \equiv a_y = p_0 - x' \qquad q \equiv a_x = q_0 + (3x + 2y') \\ &\text{From the definition of } \mathbf{W} \quad W_x = -z \qquad W_y = -z' \end{aligned}\right\} \quad (14)$$

Besides the components of relative position and velocity, these expressions contain three elements of the nominal orbit which are connected with the position of P_0 on this orbit. They can be computed from the true anomaly and the eccentricity as follows:

$$L_0 = M - v = -2e \sin v + \tfrac{1}{2}e^2 \sin 2v \quad (15)$$

to the second order, and

$$p_0 = -e_0 \sin v_0 \qquad q_0 = e_0 \cos v_0 \quad (16)$$

The quantities x,y,z,x',y',z' follow from the actual components of relative position and velocity by the definition, Eqs. (3). In particular, one has for x'

$$x' = \frac{r_0 \dot{\xi} - \xi \dot{r}_0}{(1 - e_0^2)^{1/2}} \quad (17)$$

and similarly for y' and z'.

Relative Motion of Two Points in Close Elliptic Orbits

Using the components of velocity in Eq. (2), the Lagrangian is

$$L = \tfrac{1}{2}[(\dot{r}_0 + \dot{\xi} - \eta\dot{v}_0)^2 + (r_0\dot{v}_0 + \xi\dot{v}_0 + \dot{\eta})^2 + \dot{\zeta}^2] - U$$

where

$$U = -\{(r_0 + \xi)^2 + \eta^2 + \zeta^2\}^{-1/2}$$

Since the motion in the nominal orbit is assumed known, use may be made of the equations of motion to express certain relations between r_0 and $\dot{v}_0$: $\ddot{r}_0 - r_0\dot{v}_0^2 = -(1/r_0^2)$ and $2\dot{r}_0\dot{v}_0 + r_0\ddot{v}_0 = 0$.

The equations of relative motion are thus

$$\ddot{\xi} - 2\dot{\eta}\dot{v}_0 - \xi\dot{v}_0^2 - \eta\ddot{v}_0 = 2\xi/r_0^3$$

$$\ddot{\eta} + 2\dot{\xi}\dot{v}_0 - \eta\dot{v}_0^2 + \xi\ddot{v}_0 = -\eta/r_0^3$$

$$\ddot{\zeta} = -\zeta/r_0^3$$

where only those parts of $\partial u/\partial\xi$, $\partial u/\partial\eta$, and $\partial u/\partial\zeta$ are used which are linear in the variables. This limits the validity of this analysis to the case in which the relative distance is "small" compared to the orbital radius.

Introducing again the nondimensional variables x,y, and z [Eq. (3)] and the true anomaly of the reference point as independent variable, the equations of motion are

$$x'' - 2y' = 3x/r_0^3\dot{v}_0^2$$

$$y'' + 2x' = 0$$

$$z'' = -z/r_0^3\dot{v}_0^2$$

The second of these may be integrated immediately and the result of this may be substituted in the first equation. Thus, one has two independent second-order equations. If, finally, use is made of the known elliptic motion of the reference point to rewrite the right-hand side

$$1/r_0^3\dot{v}_0^2 = (1 + e_0 \cos v_0)^{-1} = 1 - e_0 \cos v_0 + \tfrac{1}{2}e_0^2(1 + \cos 2v_0) + \ldots \quad (18)$$

the equations become

$$x'' + x = 2C - 3xe_0 \cos v_0 \quad (19)$$

$$z'' + z = e_0 z \cos v_0 \quad (20)$$

For the present purpose it is sufficient to include only the first power of the nominal eccentricity in Eq. (18) and to further specialize the problem to the case in which $v_0(t = 0) = 0$. After a solution for x is obtained, y follows from

$$y = Cv_0 + D - 2\int_0^{v_0} x dv \quad (21)$$

Equations (19) and (20) may be solved by a simple perturbation technique, based on assuming a solution in the form

$$x = \sum_{i=1}^{\infty} e_0{}^i x_i \qquad z = \sum_{i=1}^{\infty} e_0{}^i z_i$$

To the first order in e_0, the result is

$$x = 2C + A \cos v_0 + B \sin v_0 + e_0(-\tfrac{3}{2}A + A \cos v_0 - B \sin v_0 + \tfrac{1}{2}A \cos 2v_0 + \tfrac{1}{2}B \sin 2v_0 - 3Cv_0 \sin v_0)$$

$$y = D - 3Cv_0 - 2A \sin v_0 + 2B \cos v_0 + e_0(3Av_0 - 2A \sin v_0 + 2B \cos v_0 - \tfrac{1}{2} A \sin 2v_0 + \tfrac{1}{2}B \cos 2v_0 + 6C \sin v_0 - 6v_0 \sin v_0)$$

$$z = E \cos v_0 + F \sin v_0 + e_0[\tfrac{1}{2}E - \tfrac{1}{6}E \cos v_0 - \tfrac{1}{3}E \cos 2v_0 + \tfrac{1}{3}F \sin v_0 - \tfrac{1}{6}F \sin 2v_0]$$

Note the appearance of "secular" terms with v_0, $v_0 \cos v_0$, and $v_0 \sin v_0$ and the appearance of higher harmonics in the terms that have e_0 as factor. The secular terms are of course unavoidable since they indicate the continuously growing relative distance between two points in close orbits of slightly different period. It is interesting to note that no terms with $v_0{}^2$ appear, even if the solution is carried out to include $e_0{}^2$. It is not surprising that the integration constant C is closely connected with the semimajor axis; it is, on the other hand, somewhat surprising that very simple relations exist between the other integration constants and elliptic parameters. In terms of the initial condition of relative position and velocity, the integration constants are

$$A = -(3x(0) + 2y'(0)) \qquad D = y(0) - 2x'(0)$$
$$B = x'(0) \qquad E = z(0)$$
$$C = 2x(0) + y'(0) \qquad F = z'(0)$$

A comparison with Eqs. (14) shows that the integration constants are precisely the changes in the orbital elements at least as far as first-order terms in the relative positions and velocities are concerned.

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A Note on Lunar Librations

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Although it is considered that gravity-gradient torque is insufficient by itself for attitude control of artificial satellites, the mechanically equivalent phenomenon of lunar librations is nevertheless of great technical interest. Recent studies of the satellite problem disclose basic features of the motion, equally valid for the moon, but not elucidated in the specialized literature on that subject. Aside from the fundamental interest in a classic problem and possible new basis for interpretation of amassed observational data, the results are important for the newer stability analysis.

The fact that the moon persistently presents the same face toward earth, enunciated more than two centuries ago as Cassini's first law, stimulated researches by Lagrange, Laplace, and many others. Departures from this idealized motion, termed physical librations, are of such small amplitude as to have thwarted all attempts to measure them astronomically, a fact the more remarkable in view of the nearly perfect symmetry of lunar mass distribution. From the purely physical viewpoint, nothing less than mathematical proof is needed to lend plausibility to an obvious fact so much at variance with intuition.

Librational motion about a mass center itself in nonuniform motion is described by Euler's equations of rigid-body motion, extended to include effects of relative motion. This system of three coupled partial differential equations for the lunar motion, strictly nonlinear, has classically been treated by studying motion slightly perturbed from equilibrium and by further limiting the analysis to the longitudinal motion that is then uncoupled.[3] The remaining two modes, termed physical libration in inclination and physical libration in node, have meanwhile been essentially neglected. Although these two strongly coupled modes appear at first sight to be the most formidable ones from the mathematical standpoint, it will now be shown that important properties of these modes are revealed by applying directly the more detailed treatments given in satellite studies. In addition, these characteristics strongly suggest that the traditional preference for isolating longitudinal motion was an unfortunate choice made long ago and not corrected by later workers.

Free lunar librations for idealized Keplerian motion in a circular orbit are governed by equations given in Ref. 1; with unimportant changes of notation to conform with standard usage in the literature on that subject, these are

$$\ddot{\alpha} + \Omega^2\left(\frac{C-B}{A}\right)\alpha - \Omega\left(1 - \frac{C-B}{A}\right)\dot{\beta} = 0 \quad (1)$$

$$\ddot{\beta} + 4\Omega^2\left(\frac{C-A}{B}\right)\beta + \Omega\left(1 - \frac{C-A}{B}\right)\dot{\alpha} = 0 \quad (2)$$

$$\ddot{\gamma} + 3\Omega^2\left(\frac{B-A}{C}\right)\gamma = 0 \quad (3)$$

Here A,B,C and α,β,γ denote, respectively, principal inertia moments and small angular displacements from equilibrium for nodal (i.e., earth-pointing), inclination (i.e., moon latitude), and longitudinal components of the physical libration, and Ω is lunar orbital angular speed. The nearly symmetrical mass distribution is shown by the smallness of the dimensionless inertia differences, which have numerical values given by (see, e.g., Ref. 2)

$$(C - A)/C = 0.000627 \qquad (B - A)/C = 0.000118 \quad (4)$$

Denoting these for convenience by ϵ_3 and ϵ_1, respectively, the third one of the differences that appear in the system of equations, denoted by ϵ_2, is closely obtained as the difference $\epsilon_3 - \epsilon_1$; these three quantities then satisfy the important inequalities

$$0 < \epsilon_1 < \epsilon_2 < \epsilon_3 \quad (5)$$

Equation (3) shows that longitudinal libration is uncoupled from the other modes, with period of free oscillation inversely proportional to the square root of ϵ_1; its numerical value is about 53 months. This is the part of the motion examined theoretically and sought unsuccessfully through observations started by Bessel more than 100 years ago. Principal attention in modern times has centered around the forced motion resulting from solar attraction and orbital ellipticity.

Physical librations in node and inclination, described by Eqs. (1) and (2), are obviously strongly coupled by the first derivative terms of order unity. Each equation admits harmonic solutions, and it is found that physical libration in inclination "leads" the nodal motion with a phase angle nearly equal to 90°. Of perhaps even greater importance from standpoint of observation is the fact that one of the two periods of free motion is very much smaller than the other, small even when compared with the period of free longitudinal

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